

# Interwoven basin structures of double logistic map at the edge of chaos

Gyu-Seung Shin\*

College of Electronics and Information, Institute of Natural Sciences,  
Kyung Hee University, Yongin, Kyonggi 449-701, South Korea

(Dated: February 8, 2008)

The carpeting regularity of basin structures for simply doubled logistic map is studied. Examining the elementary structures of which global basin structures are composed, we found them to have a impressive interrelation which we call *all in one and one in all*. Curves appearing in the basins and their expressions are also discussed.

PACS numbers: 05.45.-a

We occasionally come across some patterns where several geometrical structures are interwoven exquisitely. Those structures seem to have very curious interrelations such that they are interwoven into others repeatedly over a wide range of scale. Consequently, a small section of the structure may contain information about the whole. Interwoven patterns and their regularity deserve to be objects of aesthetical appreciation and scientific research. Prominent instances would be those patterns appearing in Mandelbrot sets and Julia sets [1]. Also, dynamic systems with multistability show various features of interwovenness in basins of attraction, such as fractal [2, 3, 4, 5], riddled [5, 6, 7, 8, 9] and intermingled [10] basin boundaries. Recently they became objects of intense investigations with related subjects including synchronization [9, 11] and noise effects [4, 8, 12].

In this paper, geometrical regularities of basin structures for two-dimensional simply doubled logistic map at the edge of chaos is investigated. These basin structures are found to have a regularity which shows global interwovenness and overall interrelation.

Simply doubled logistic map is

$$\begin{cases} x_{t+1} \\ y_{t+1} \end{cases} = \begin{cases} \mu x_t(1 - x_t) \\ \mu y_t(1 - y_t) \end{cases}. \quad (1)$$

It is a very simple two-dimensional map which is physically trivial. Nevertheless, it shows very regular interwovenness which may have referential significances for other interwoven structures.

In the period-2 region, the map has two attractors “in-phase”,  $(x_1, x_1) \leftrightarrow (x_2, x_2)$  and “out-of-phase”,  $(x_1, x_2) \leftrightarrow (x_2, x_1)$  where  $x_1$  and  $x_2$  are cyclic points of the logistic map,  $x_{t+1} = F_\mu(x_t) = \mu x_t(1 - x_t)$ . Likewise, the Eq. (1) has  $2^p$  attractors in the  $2^p$ -cycle region with phases,

$$y_t = x_{t+m} \quad \text{for} \quad m = 0, 1, 2, \dots, 2^p - 1.$$

With  $\mu = 3.4$  in period-2 region, the map has two basins of attraction as shown in Fig. 1, where dark regions are

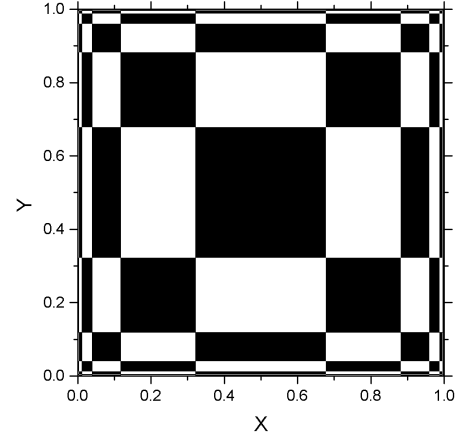


FIG. 1: Two basins of attractions at  $\mu = 3.4$ . Dark regions are the basin of the “in-phase” attractor while white ones are for the “out-of-phase” attractor.

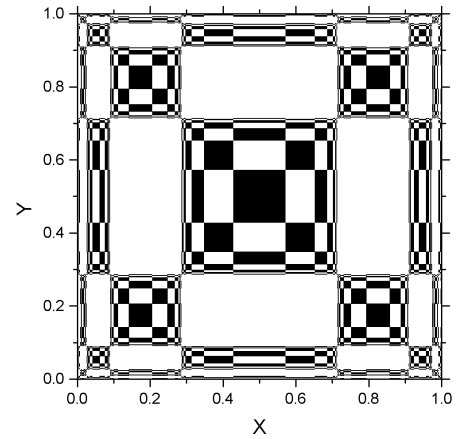


FIG. 2: The basin of “in-phase” attractor (dark regions) with  $\mu = 3.54$ (period-4). Basins of the other three attractors will occupy white regions.

the basin of the “in-phase” attractor while white rectangles are that of the “out-of-phase” attractor.

Divisions of these rectangular regions can be given with straight lines,  $x = x_n$ ,  $x = 1 - x_n$ ,  $y = x_n$ , and  $y = 1 - x_n$

\*Electronic address: shings@khu.ac.kr

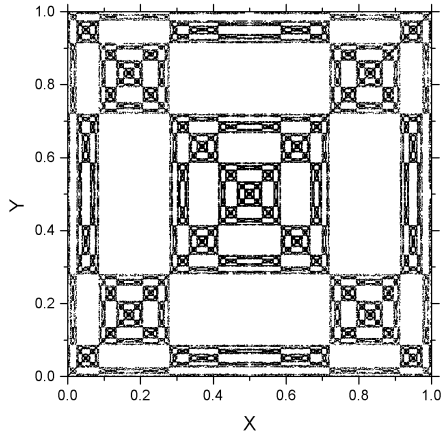


FIG. 3: The basin of the “in-phase” attractor with  $\mu = 3.57$  very near to the edge of chaos.

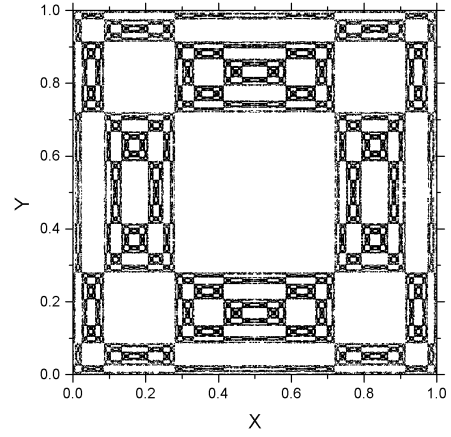


FIG. 6: The basin of an attractor with phase  $y_t = x_{t+3}$ .

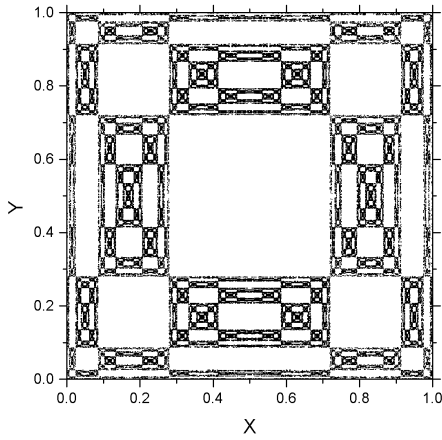


FIG. 4: The basin of an attractor with phase  $y_t = x_{t+1}$ .

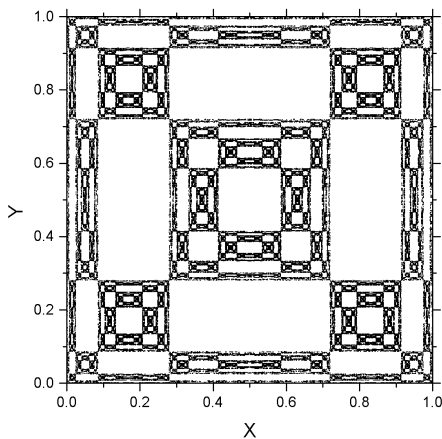


FIG. 5: The basin of an attractor with phase  $y_t = x_{t+2}$ .

with  $x_n$ 's as

$$x_n = \max\{x | F_\mu^{(n-1)}(x) = F_\mu^{(n)}(x)\} \quad \text{for } n = 1, 2, 3, \dots \quad (2)$$

The basin of the “in-phase” attractor with  $\mu = 3.54$  in the period-4 region is shown in Fig. 2 with dark regions, while white regions will be occupied by the other three basins. For the convenience of later discussions, let us designate rectangles white or dark in Fig. 1 as “primary”, and inner rectangles in “primary” ones as seen in Fig. 2 as “secondary” rectangles. Actually there are an infinite number of “primary” rectangles with a wide range of sizes and proportions in Fig. 1. Consequently, most of them have infinitesimal area.

Our main concern is basin structures with  $\mu = \mu_\infty$ , on the edge of chaos where there exist an infinite number of attractors. Among them, a few basins of attractors with small phase differences between  $x$  and  $y$  are selected and presented in Figs. 3 - 6 ( $y_t = x_t$ ,  $y_t = x_{t+1}$ ,  $y_t = x_{t+2}$ ,  $y_t = x_{t+3}$  in sequence). In these Figures, we can see repeated appearances of some patterns in various sizes with winding curves, thus, one might suspect the existence of a carpeting regularity ruling all basin structures. We may ask how the patterns of those basin carpets are composed, and what those curves are.

A simple and effective way to answer the first question is to elucidate the overall interrelation among basin structures rather than enumerate each of them. Examining every “primary” rectangle in global basin structures, its pattern is found to have correspondence with a global basin structure. It is needed to magnify “primary” rectangles whose sizes are too small for direct examination. The result is summarized in Fig. 7 and Fig. 8 which represent central parts of  $[2m]$  and  $[2m+1]$ , global basin structures of two attractors with phases  $y_t = x_{t+2m}$  and  $y_t = x_{t+(2m+1)}$ . Here we let a symbol  $[m]$  denote the pattern of the global basin structure for an attractor with phase  $y_t = x_{t+m}$ , and the letter  $m$  in Fig. 7 and Fig. 8 has the same meaning. We should remark that  $[m]$  rep-

|     |     |     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| m   |     | m+1 |     | m+2 |     | m+1 |     | m   |
|     | m   |     | m+1 |     | m+1 |     | m   |     |
| m-1 |     | m   |     | m+1 |     | m   |     | m-1 |
|     | m-1 |     | m   |     | m   |     | m-1 |     |
| m-2 |     | m-1 |     | m   |     | m-1 |     | m-2 |
|     | m-1 |     | m   |     | m   |     | m-1 |     |
| m-1 |     | m   |     | m+1 |     | m   |     | m-1 |
|     | m   |     | m+1 |     | m+1 |     | m   |     |
| m   |     | m+1 |     | m+2 |     | m+1 |     | m   |

FIG. 7: Global pattern of basin for an attractor with phase  $y_t = x_{t+2m}:[2m]$ . Squares denotes “primary” rectangles which actually do not have equal areal sizes and proportions. The central “primary” square is occupied by  $[m]$ .

|     |     |     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|     | m+1 |     | m+2 |     | m+2 |     | m+1 |     |
| m   |     | m+1 |     | m+2 |     | m+1 |     | m   |
|     | m   |     | m+1 |     | m+1 |     | m   |     |
| m-1 |     | m   |     | m+1 |     | m   |     | m-1 |
|     | m-1 |     | m   |     | m   |     | m-1 |     |
| m-1 |     | m   |     | m+1 |     | m   |     | m-1 |
|     | m   |     | m+1 |     | m+1 |     | m   |     |
| m   |     | m+1 |     | m+2 |     | m+1 |     | m   |
|     | m+1 |     | m+2 |     | m+2 |     | m+1 |     |

FIG. 8: Structure of basin with phase  $y_t = x_{t+(2m+1)}:[2m+1]$ . The central square is vacant and the nearest rectangles are occupied by  $[m]$  and  $[m+1]$ .

resents only the pattern of the basin structure and not its exact proportion or size, and all “primary” rectangles are presented as squares in Fig. 7 and Fig. 8 in spite of differences in area and proportion.

The rectangles are occupied in a checker pattern and the largest central “primary” square as in Figs. 3 - 6 is oriented at the center of Fig. 7 or Fig. 8. In  $[2m]$  (Fig. 7), the central square and those along the diagonals are occupied by  $[m]$ , this is followed by  $[m-1]$ ,  $[m-2]$ , etc. moving horizontally away from the diagonals and by  $[m+1]$ ,  $[m+2]$ , etc. moving vertically away from the diagonals. For  $[2m+1]$ , the central square is empty and the nearest rectangles are occupied with  $[m]$  and  $[m+1]$  as shown in Fig. 8. Accordingly, global patterns of Figs. 3 - 6 can be symbolized as  $[0]$ ,  $[1]$ ,  $[2]$  and  $[3]$  respectively. The same rule will hold for the relation between “primary” rectangles and “secondary” ones and also for

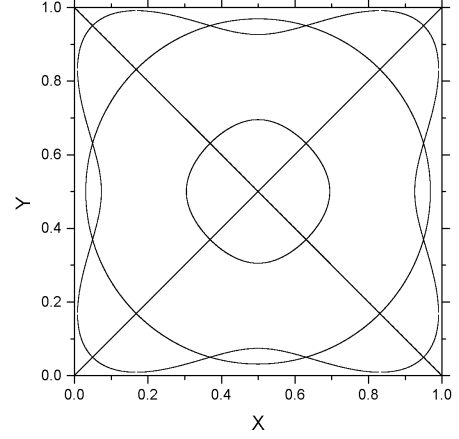


FIG. 9: Curves appearing in the Fig. 3. The outermost circle is expressed by Eq. (3) and Eq. (4) is the expression for the winding loop and inner circlelike loop.

rectangles of further levels.

The pattern of basin carpet  $[m]$  can be defined by  $\{\dots b_3 b_2 b_1\}$ , the binary notation of an integer number  $m$ , as the last digit  $b_1$  determines occupations of “primary” rectangles and  $b_2$  arranges “secondary” ones. With their binary variations, these basin patterns remind us of the T’ai Chi (Tai Ji) diagram in I jing (Yijing, Book of Changes) which is the central frame in chinese cosmology [13].

We can see the emergence of a large number of curves which are crossing each other in Figs. 3 - 6. The appearance of curves seems to be a consequence of the edge of chaos. In Fig. 3, noticeable ones are diagonal lines, a circular shape and more winding curves. We attempt to express them respectively as

$$\begin{aligned}\bar{y} &= \bar{x}, \\ \bar{y} &= -\bar{x},\end{aligned}$$

$$\bar{x}^2 + \bar{y}^2 + A_1 = 0, \quad (3)$$

and

$$\bar{x}^4 + \bar{y}^4 + B_1 \bar{x}^2 + B_1 \bar{y}^2 + B_2 = 0, \quad (4)$$

with proper constants which can be expressed with  $x_n$ ’s of Eq. (2) as

$$\begin{aligned}A_1 &= B_1 = -\bar{x}_1^2 - \bar{x}_2^2, \\ B_2 &= \bar{x}_1^2 \bar{x}_2^2 + \bar{x}_1^2 \bar{x}_3^2 + \bar{x}_2^2 \bar{x}_3^2 - \bar{x}_3^4.\end{aligned}$$

Here we used  $\bar{x}$ ,  $\bar{y}$  and  $\bar{x}_n$  as  $\bar{x} \equiv x - \frac{1}{2}$ ,  $\bar{y} \equiv y - \frac{1}{2}$  and  $\bar{x}_n \equiv x_n - \frac{1}{2}$  for simplicity of expressions. Curves of these expressions are depicted in Fig. 9, they show exact coincidences with the curves in Fig. 3.

In this paper, we have studied orderly interwoven patterns in basins of double logistic map. Interwovenness

is beautiful from an aesthetic viewpoint and certainly has some geometrical regularities. But for the most part they are emergent and mystic, so, direct apprehension of them is not easy. We have dealt with interwoven patterns which are exceptionally regular, with a definite global order. This regularity may have referential significance for other interwoven structures, as well as for basin structures of coupled maps. The order in these patterns may be visible facets of an exquisite regularity for which there remains much room for further investigation. We see that these basin structures have impressive interrelations which might be called *all in one and one in all*, all the basin structures are contained in a basin which takes part in building up the all *vice versa* over a wide range of

scale. *This interrelation suggests an interesting idea to us about the wholeness and the elementariness.* Interwovenness may be ubiquitous, and is an interesting aspect of nonlinear phenomena in nature. Clarification of the regularity might be a way to understand the nonlinear features.

### Acknowledgments

The author acknowledges a critical reading of the manuscript and also fruitful discussions with Michael Walker.

- 
- [1] H.-O. Peitgen and D. H. Richter, *The Beauty of Fractals: Images of Complex Dynamical Systems* (Springer-Verlag, Berlin, 1986). B. B. Mandelbrot, *The Fractal Geometry of Nature*, (W. H. Freeman, New York, 1982). H. Lauwerier, *Fractals: Endlessly Repeated Geometric Figures*, (Princeton University Press, Princeton, NJ, 1991). M. Schroeder, *Fractals, Chaos, Power Laws*, (W. H. Freeman, New York, 1991).
  - [2] C. Grebogi, S. W. McDonald, E. Ott, and J. A. Yorke, Phys. Lett. A **99**, 415 (1983). C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. Lett. **50**, 935 (1983). F. T. Arecchi, R. Badii and A. Politi, Phys. Rev. A **29**, 1006 (1984).
  - [3] F. C. Moon and G. X. Li, Phys. Rev. Lett. **55**, 1439 (1985). K. Kacperski and J. A. Holyst, Phys. Rev. E **60**, 403 (1999).
  - [4] E. G. Gwinn and R. M. Westervelt, Phys. Rev. Lett. **54**, 1613 (1985). M. Iansiti, Q. Hu, R. M. Westervelt, and M. Tinkham, Phys. Rev. Lett. **55**, 746 (1985).
  - [5] Y. -C. Lai, Phys. Rev. E **62**, 4505 (2000).
  - [6] A. Pikovsky and P. Grassberger, J. Phys. A **24**, 4587 (1991). J. C. Alexander, J. A. Yorke, Z. You and I. Kan, Int. J. Bifurcation Chaos Appl. Sci. Eng. **2**, 795 (1992). J. C. Sommerer and E. Ott, Nature (London) **365**, 136 (1993). E. Ott, J. C. Sommerer, J. C. Alexander, I. Kan, and J. A. Yorke, Phys. Rev. Lett. **71**, 4134 (1993). J. F. Heagy, T. L. Carroll, and L. M. Pecora, Phys. Rev. Lett. **73**, 3528 (1994).
  - [7] Y. -C. Lai, C. Grebogi, J. A. Yorke, and S. C. Venkataramani, Phys. Rev. Lett. **77**, 55, (1996). K. Kaneko, Phys. Rev. Lett. **78**, 2736 (1997). Y. L. Maistrenko, V. L. Maistrenko, A. Popovich, and E. Mosekilde, Phys. Rev. E **57**, 2713 (1998). M. Woltering and M. Markus, Phys. Rev. Lett. **84**, 630 (2000).
  - [8] Y. -C. Lai and C. Grebogi, Phys. Rev. Lett. **77**, 5047 (1996).
  - [9] O. V. Sosnovtseva, A. G. Balanov, T.E. Vadivasova, V. V. Astakhov, and E. Mosekilde, Phys. Rev. E **60**, 6560 (1999).
  - [10] Y. -C. Lai and C. Grebogi, Phys. Rev. E **52**, R3313 (1995). J. C. Sommerer and E. Ott, Phys. Lett. A **214**, 243 (1996). M. Schmick, E. Goles, and M. Markus, Phys. Rev. E **62**, 397 (2000).
  - [11] V. Astakhov, A. Shabunin, T. Kapitaniak, and V. Anishchenko, Phys. Rev. Lett. **79**, 1014 (1997). H. L. Yang, Phys. Rev. E **63**, 026213 (2001). A. Krawiecki and S. Matyjaskiewicz, Phys. Rev. E **64**, 036216 (2001).
  - [12] L. Billings, E. M. Bollt, and I. B. Schwartz, Phys. Rev. Lett. **88**, 234101, (2002).
  - [13] F. Capra, *The Tao of Physics*, (Wildwood House, London, 1975).